

# Universality in the One-Dimensional Self-Organized Critical Forest-Fire Model\*

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We modify the rules of the self-organized critical forest-fire model in one dimension by allowing the fire to jump over holes of  $\leq k$  sites. An analytic calculation shows that not only the size distribution of forest clusters but also the size distribution of fires is characterized by the same critical exponent as in the nearest-neighbor model, i.e. the critical behavior of the model is universal. Computer simulations confirm the analytic results.

## I. Introduction

Some years ago, Bak, Tang, and Wiesenfeld introduced the *sandpile model* which evolves into a critical state irrespective of initial conditions and without fine tuning of parameters [1]. Such systems are called *self-organized critical* (SOC) and exhibit power-law correlations in space and time. The concept of SOC has attracted much interest since it might explain the origin of *fractal structures* and *1/f-noise*. Other SOC models e.g. for earthquakes [2, 3] or the evolution of populations [4, 5] have been introduced since then, improving our understanding of the mechanisms leading to SOC. Recently, a forest-fire model has been introduced which can be viewed as a model for excitable media [6, 7]. It becomes self-organized critical when time scales are separated [8]. In one dimension, the critical exponents could be determined analytically, thus proving the possibility of SOC in nonconservative systems [9].

Analogous to critical phenomena in equilibrium phase transitions, it is expected that the values of the critical exponents depend only on few macroscopic properties of the system as dimension, conservation laws and symmetries, i.e. the critical behavior of the model is *universal*. Computer simulations of the forest-fire model for different lattice symmetries and for a modification with immune trees show indeed universal behavior [10, 11], but so far this observation has no analytic foundation.

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In this paper, we show by analytic means that the critical exponents in the one-dimensional forest-fire model are universal when the fire is allowed to jump over holes up to a given size. In Sect. II, we introduce the rules of the model. In Sect. III, we give a short review of the analytic solution in [9]. In Sect. IV, we calculate the critical exponent for the fire size distribution when the fire is allowed to jump over holes of up to  $k$  sites. The values of the exponents are confirmed by computer simulations. Finally, we summarize our results.

## II. The Model

The forest-fire model is a stochastic cellular automaton which is defined on a hypercubic lattice with  $L^d$  sites. In this paper, we consider only the one-dimensional case  $d=1$ . Each site is occupied by a tree, a burning tree, or it is empty. During one time step, the system is parallelly updated according to the following rules

- burning tree  $\rightarrow$  empty site,
- tree  $\rightarrow$  burning tree, if at least one neighbor in a distance  $\leq k+1$  is burning,  $k=0, 1, 2, \dots$ ,
- tree  $\rightarrow$  burning tree with probability  $f$ , if no neighbor is burning,
- empty site  $\rightarrow$  tree with probability  $p$ .

Starting with arbitrary initial conditions, the system approaches after a transition period a steady state the properties of which depend only on the parameter values. We always assume that the lattice is so large that no finite size effects occur. The steady state is self-organized critical if the parameters satisfy a dou-

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ble separation of time scales

$$f \ll p \ll f/p. \quad (1)$$

The first inequality means that many trees grow between two lightning strokes and therefore large forest clusters and fires occur. The second inequality means that even large forest clusters burn down before new trees grow at their edge. Under these conditions, the size distributions of forests and fires obey power laws as we shall see below.

### III. Analytic Solution for $k = 0$

In the case  $k = 0$ , where the fire is stopped by any empty site, i.e. just jumps to nearest neighbors, many properties of the model have been derived analytically in [9]. Before proceeding to general  $k$ , we give an illustrative derivation of these results.

The mean number of trees destroyed by a lightning stroke is

$$\bar{s} = (f/p)^{-1} (1 - \varrho)/\varrho, \quad (2)$$

where  $\varrho$  is the mean forest density in the steady state [8].

Let  $n(s)$  be the mean number of forest clusters of  $s$  trees, divided by the number of sites  $L$ .  $n(s)$  will be shown to obey a power law

$$n(s) \propto s^{-\tau} \quad (3)$$

for clusters smaller than a cutoff

$$s_{\max} \propto (f/p)^{-\lambda}, \quad (4)$$

besides of possible logarithmic corrections. The probability that lightning strikes a forest cluster of size  $s$  is proportional to  $s n(s)$ . Since the fire is stopped by any empty site, the size distribution of fires is also proportional to  $s n(s)$ .

In order to derive the size distribution of fires, consider a string of  $n \ll p/f$  sites. This string is too short for two trees to grow during the same time step. Lightning does not strike this string before all of its trees are grown. Since we are always interested in the limit  $f/p \rightarrow 0$ , the following considerations remain valid even for strings of a very large size. Starting with a completely empty state, the string passes through a cycle which is illustrated in Figure 1. During one time step, a tree grows with probability  $p$  on any site. After some time, the string is completely occupied by trees. Then the forest in the neighborhood of our string will

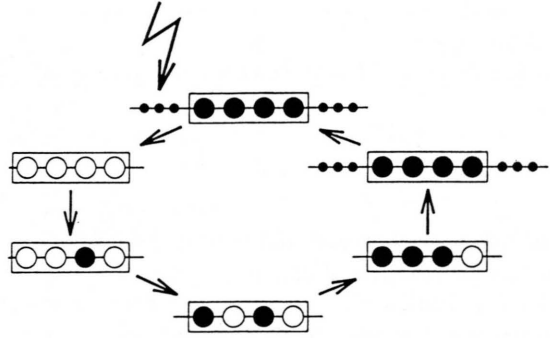


Fig. 1. Dynamics on a string of  $n = 4$  sites. Trees are black, empty sites are white.

also be quite dense. The forest on our string is part of a forest cluster which is much larger than  $n$ . Eventually that cluster becomes so large that it is struck by lightning with a nonvanishing probability. Then the forest cluster burns down, and the string again becomes completely empty. (For a rigorous justification of the neglect of lightning strokes on the  $n$ -string leading to random growth of trees, we refer the reader to [9].)

This consideration allows us to write down rate equations for the states of the string. In the steady state, each configuration of trees is generated as often as it is destroyed. Let  $P_n(m)$  be the probability that our string is occupied by  $m$  trees. Each configuration which contains the same number of trees has the same probability. A configuration of  $m$  trees is destroyed when a tree grows at one of the empty sites, and is generated when a tree grows in a state consisting of  $m-1$  trees. The completely empty state is generated when a dense forest burns down. Since all trees on our string burn down simultaneously, this happens each time when a given site of our string is on fire. This again happens as often as a new tree grows at this given site, i.e. with probability  $p(1-\varrho)$  per time step. We therefore have the following equations (which have been derived more formally in [9])

$$p n P_n(0) = p(1-\varrho),$$

$$p(n-m) P_n(m) = p(n-m+1) P_n(m-1) \quad \text{for } m \neq 0, n.$$

We conclude

$$\begin{aligned} P_n(m) &= (1-\varrho)/(n-m) \quad \text{for } m < n, \\ P_n(n) &= 1 - (1-\varrho) \sum_{m=0}^{n-1} 1/(n-m) \\ &= 1 - (1-\varrho) \sum_{m=1}^n 1/m. \end{aligned} \quad (5)$$

A forest cluster of size  $s$  is a configuration of  $s$  neighboring trees with an empty site at each end. The size distribution of forest clusters consequently is

$$n(s) = \frac{P_{s+2}(s)}{\binom{s+2}{s}} = \frac{1-\varrho}{(s+1)(s+2)} \simeq (1-\varrho)s^{-2}. \quad (6)$$

This is a power law with the critical exponent  $\tau = 2$ . The size distribution of fires is  $\propto sn(s) \propto s^{-1}$ .

There is a characteristic length  $s_{\max}$  where the power law  $n(s) \propto s^{-2}$  breaks down. We calculate  $s_{\max}$  from the condition that a string of size  $n \leq s_{\max}$  is not struck by lightning until all trees are grown. When a string of size  $n$  is completely empty at time  $t = 0$ , it will be occupied by  $n$  trees after

$$T(n) = (1/p) \sum_{m=1}^n 1/m \simeq \ln(n)/p$$

timesteps on an average. The mean number of trees after  $t$  timesteps is

$$m(t) = n[1 - \exp(-pt)].$$

The probability that lightning strikes a string of size  $n$  before all trees are grown is

$$f \sum_{t=1}^{T(n)} m(t) \simeq (f/p) n(\ln(n) - 1) \simeq (f/p) n \ln(n).$$

We conclude

$$s_{\max} \ln(s_{\max}) \propto p/f \quad \text{for large } p/f, \quad (7)$$

leading to  $\lambda = 1$ .

Next we determine the relation between the mean forest density  $\varrho$  and the parameter  $f/p$ . The mean forest density is given by

$$\begin{aligned} \varrho &\simeq \sum_{s=1}^{s_{\max}} sn(s) \\ &= (1-\varrho) \sum_{s=1}^{s_{\max}} \frac{s}{(s+1)(s+2)} \\ &\simeq (1-\varrho) \ln(s_{\max}). \end{aligned}$$

Thus

$$\frac{\varrho}{1-\varrho} \simeq \ln(s_{\max}) \simeq \ln(p/f) \quad \text{for large } p/f. \quad (8)$$

The forest density approaches the value 1 at the critical point. This is not surprising since no infinitely large cluster exists in a one-dimensional system as long as the forest is not completely dense. Combining (6) and (8), we obtain the final result for the cluster-size

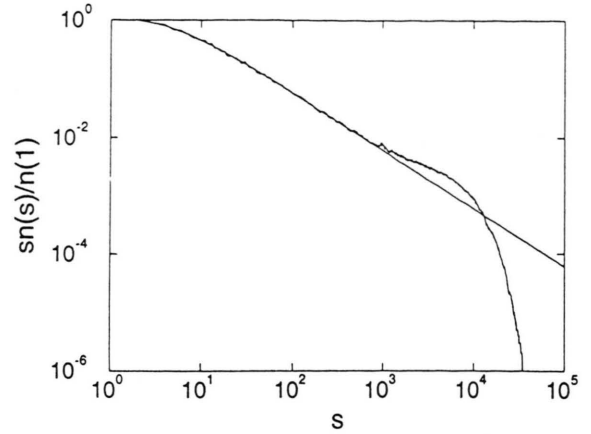


Fig. 2. Size distribution of the fires for  $f/p = 1/25000$  and  $L = 2^{20}$ . The smooth line is the theoretical result, which is valid for cluster sizes  $\leq s_{\max}$ .

distribution near the critical point

$$n(s) \simeq \frac{1}{(s+1)(s+2) \ln s_{\max}} \quad \text{for } s < s_{\max} \quad (9)$$

with  $s_{\max}$  given by (7).

The size distribution  $sn(s)$  of the fires has also been determined by computer simulations. The result is shown in Figure 2. It agrees perfectly with (6) in the region  $s < s_{\max}$ .

#### IV. Universality of the Critical Exponents

We now allow the fire to spread to trees up to a distance  $k+1$  from a burning tree, as given by the second rule above. The fire jumps over holes of up to  $k$  empty sites, but is stopped by holes of more than  $k$  sites. Consequently a fire no longer destroys just a single forest cluster, but it may also destroy several clusters which are separated by holes of  $\leq k$  sites. The size distribution of fires therefore is no longer given by  $sn(s)$ . In this section, we will show that the critical exponent which describes the size distribution of fires is still 1 indicating its universality.

We consider a string of  $n \gg k$  sites which is empty in the beginning. After  $t$  timesteps, it contains a hole of size  $k+1$  with the probability

$$n(1-p)^{(k+1)t} (1-(1-p)^t)^2 \simeq ne^{-(k+1)t p}$$

for small  $p$  but large  $pt$ . The time after which there are no holes larger than  $k$  therefore is proportional to

$\ln(n)/p(k+1)$ . This time becomes very long for large values of  $n$ , and consequently the forest is very dense at the moment where all holes larger than  $k$  have disappeared. The critical forest density therefore is still  $\varrho_c = 1$ . As long as  $\varrho < 1$ , there is a nonvanishing probability that a hole larger than  $k$  occurs, and the fire cannot spread indefinitely.

As in the previous section, we choose  $n$  so small that the string is not struck by lightning before all empty sites have disappeared, i.e.  $n < s_{\max}$ . In the limit  $f/p \rightarrow 0$ ,  $s_{\max}$  diverges, and the string can be very large. The dynamics on our string are exactly the same as before (Fig. 1), and (5) to (7) are still valid. The size distribution of forest clusters smaller than  $s_{\max}$  remains unchanged, and the critical exponents  $\tau = 2$  and  $\lambda = 1$  characterizing this distribution are universal. Equation (8) for the forest density also remains the same in the limit  $f/p \rightarrow 0$  (besides of a constant which has to be added to the right-hand side of (8) but which has already been neglected before, since it is much smaller than  $\ln(p/f)$ ).

Now we calculate the size distribution  $F(m)$  of fires that destroy  $m$  trees. For  $m < s_{\max}$ , it can be derived using (5). It is proportional to  $m$  times the number of configurations which contain  $m$  trees with at least  $k+1$  empty sites at each end and no holes larger than  $k$  sites between the trees, i.e.

$$F(m) \propto m \sum_{N=0}^{k(m-1)} \frac{P_{m+N+2k+2}(m)}{\binom{m+N+2k+2}{m}} \cdot \sum_{\{l_0, \dots, l_k\}, \sum_i l_i = m-1, \sum_i i l_i = N} \frac{(m-1)!}{l_0! \cdots l_k!}. \quad (10)$$

Since we are only interested in the asymptotic power law for large  $m$ , this sum can be simplified. The main contribution comes from values  $N \ll m$  since for larger  $N$  there are only few configurations which contain no holes larger than  $k$  (see the consideration at the beginning of this section). The probability for a hole of size  $k+1$  on a string of  $m+N$  sites with  $N \ll m$  empty sites is<sup>1</sup>

$$\simeq (m-1) \frac{(m-2)^{N-k-1}/(N-k-1)!}{(m-1)^N/N!} \simeq \frac{N^{k+1}}{m^k},$$

<sup>1</sup> This result is valid only for  $N \ll m$  and is obtained as follows: There are  $m-1$  possible positions for the hole of size  $k+1$ . The remaining  $N-k-1$  empty sites can be distributed at random over the remaining  $m-2$  gaps between trees. To obtain a probability, we have to divide by all possible configurations of  $N$  empty sites on  $m-1$  gaps.

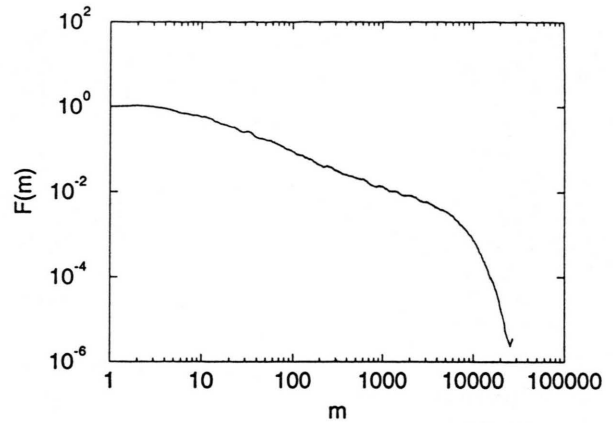


Fig. 3. Size distribution of the fires for  $f/p=1/8000$  and  $L=2^{20}$ . The fire is allowed to jump over holes of one empty site.

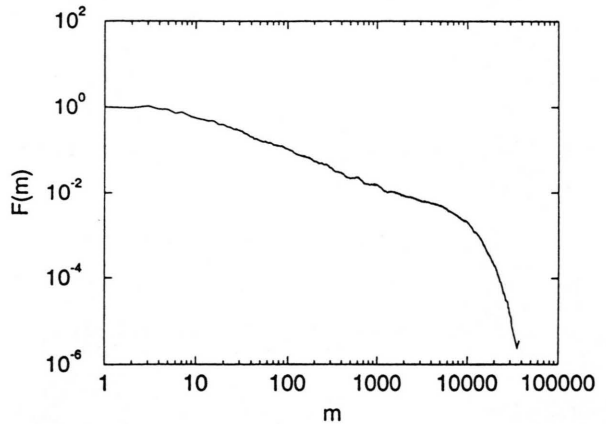


Fig. 4. Size distribution of the fires for  $f/p=1/8000$  and  $L=2^{20}$ . The fire is allowed to jump over holes of two empty sites.

from which we conclude that the first sum in (10) has a cutoff for  $N \propto m^{k/(k+1)}$ . The second sum in (10) counts the number of different configurations of  $N$  empty sites on  $m-1$  gaps with the restriction that each gap contains no more than  $k$  empty sites. For  $N$  smaller than the cutoff, the probability for holes larger than  $k$  is very small anyway, and we are therefore allowed to sum over all configurations of  $N$  empty

<sup>2</sup> This is equivalent to the number of configurations of  $m$  trees on  $N+m$  sites with a tree at both ends.

sites on  $m-1$  gaps which gives  $\binom{m+N-2}{m-2}$ .<sup>2</sup> We then obtain

$$F(m) \propto \sum_{N=0}^{m^{k/(k+1)}} \frac{m}{N+2k+2} \frac{\binom{m+N-2}{m-2}}{\binom{m+N+2k+2}{m}} \\ \simeq \sum_{N=0}^{m^{k/(k+1)}} \left(\frac{N}{m}\right)^{2k+1} \propto \frac{1}{m}. \quad (11)$$

This is a power law with an exponent 1 which is independent of  $k$ . We thus have shown that not only the exponent for the size distribution of forests but also the exponent for the size distribution of fires is universal. The form of the cutoff functions describing the behavior of these distributions on lengths larger than  $s_{\max}$ , however, is different for different values of  $k$ .

Our computer simulations confirm the analytic result. In Figs. 3 and 4, the size distribution of fires is shown for fire propagation over holes of size 1 and 2. The slope in the scaling regions is  $-1$  each time.

## V. Conclusion

In this paper we have shown by analytic means that the critical exponents of the SOC forest-fire model in one dimension show universal behavior when the range of the interaction is changed. This is additionally confirmed by computer simulations.

In two dimensions, too, computer simulations show that the model is universal under a change of the lattice symmetry. It still remains a challenge to prove universality in dimensions higher than two analytically, e.g. by the renormalization group formalism.

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